Population inversion through charge measurement using a superconducting single-electron transistor biased in the subgap regime

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We show how population inversion (PI) occurs in a two-level system (TLS) while measuring its charge using a capacitively coupled superconducting single-electron transistor (SSET), biased in the subgap regime, where the current through the SSET is carried by different cycles involving tunneling of both Cooper pairs and quasiparticles. The PI is directly associated with the resonant nature of the Cooper-pair tunneling. We also show how the SSET may strongly relax the TLS, although there is negligible current flowing through the SSET, i.e. it is turned off. The calculation of the quantum back-action noise is based on a real-time Keldysh approach. We specifically discuss the case of a Cooper Pair Box qubit with the SSET capacitively coupled as read-out device.

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The single-electron transistor (SET) is known as an extremely sensitive electrometer [1, 2] based on the Coulomb blockade[3, 4]. The SET consists of a small capacitance island, isolated from the leads by tunnel junctions. The measurement is performed by measuring the current flowing through the island, which depends strongly on the charge induced through the capacitive coupling to the charge to be measured.

With the invention of the RF-SET[5], the SET was also made fast, and operating frequencies above 100 MHz could be reached. Therefore the SET can be used in applications ranging from very sensitive charge meters and current standards[4] in which electrons are counted or pumped one by one, to read-out of quantum bits[6, 7, 8, 9] or to work as photon detectors[10].

For these sensitive applications the back-action of the SET during read-out, arising from the voltage fluctuations on the SET island, is important since it may corrupt the measurement[11, 12].

The SET may also be used in the superconducting state (SSET) and indeed such a SSET shows the state-of-the-art sensitivity of $3.2\cdot10^{-6}\mathrm{e}/\sqrt{\mathrm{Hz}}$ [13]. This SSET was biased around the gap edge, i.e. the threshold voltage for sequential quasiparticle tunneling[14], where the back-action may be analyzed in the same way as for sequential electron tunneling in the normal state [11, 12, 15].

There have also been suggestions for using the SSET biased in the subgap regime[16, 17], where the current is carried by different sequences of resonant Cooper-pair tunneling and quasiparticle tunneling, so called Josephson-Quasiparticle (JQP) cycles[1, 18].

The current noise, determining the shot-noise limited sensitivity was recently analyzed for the simplest JQP cycle[19]. This process consists of a Cooper-pair tunneling across one junction followed by two quasiparticles tunneling across the opposite junction. Clerk et al[20] analyzed a more complicated process with two Cooper pairs involved, called the Double JQP process (DJQP), which corresponds to the so called eye-feature in the SSET current-voltage characteristics, see Fig. 3b. The resonant Cooper-pair tunneling was shown to improve sensitivity.

For the DJQP it was shown that also the back-action changes qualitatively and may even induce population inversion (PI) in the measured system[20]. This shows that the back-action of the SSET indeed is very non-trivial, e.g. simply assigning a noise-temperature to this device would demand negative temperatures.

In this paper we describe the back-action in the whole subgap regime. We specifically map out where PI is possible, which also includes the simplest JQP process. Furthermore, we show where the relaxation of the measured system may be strong, even though there is negligible current flowing through the SSET, i.e. it is turned off. We choose the Single Cooper Pair box (SCB)[21] as our specific two-level system (TLS).

The Model Consider a small superconducting SET island coupled via low transparency tunnel barriers to two external superconducting leads, and coupled capacitively to a control gate and to the SCB, see Fig.1. The voltage noise on the SET island is calculated neglecting the coupling to the SCB. This approach is appropriate in the considered limit of weak SET-box coupling ($C_c \ll C_L \sim C_R \sim C_b$).

We follow the outline of Ref. [18] and model the SSET by the Hamiltonian

$$H = H_L+H_R+H_I+H_C+K+H_J+H_T = H_0+H_T, \quad (1)$$

where $H_r = \sum_{kn}\epsilon^r_{kn}a^\dagger_{krn}a_{krn}$ and $H_I = \sum_{ln}\epsilon_{ln}c^\dagger_{ln}c_{ln}$ describe the noninteracting quasiparticles in the $r \in \{L, R\}$ leads and on the island. Introducing the operators $\hat{n}_{L/R}$ for the number of charges passed from left to right in the L/R junction we write the Coulomb term $H_C = E_C(\hat{N} - n_x)^2$ where $E_C = e^2/2C_\Sigma$ is the charging energy and $\hat{N} = \hat{n}_L - \hat{n}_R$ is the operator for the excess number of charges on the island. $C_\Sigma \approx C_L + C_R$ is the island capacitance neglecting the small gate and coupling capacitances, and n_x is the fractional number of electrons

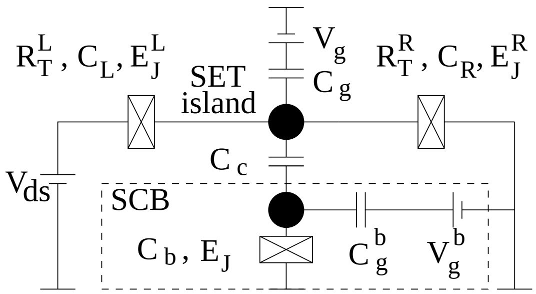

FIG. 1: Schematic figure of the superconducting single-electron transistor capacitively coupled to a two-level system, in this case a Single Cooper Pair box (SCB).

induced by the gate voltage C_gV_g/e. The work done by the biasing circuit is $K = eV_{ds}(C_R\hat{n}_L + C_L\hat{n}_R)/C_\Sigma$ and the Josephson coupling is $H_J = -\sum_r E_J^r \cos 2\Phi_r$, where $e^{i\Phi_r}$ increases n_r by one. Then we work in the eigenbasis of H_0, and treat the quasiparticle tunneling H_T as the perturbation. We use a real-time diagrammatic Keldysh technique[12, 22] to calculate the DC current and the spectral density of the SET island voltage fluctuations,

$$S_V(\omega) = \frac{e^2}{C_\Sigma^2}\int_{-\infty}^{\infty} d\tau e^{+i\omega\tau} Tr\{\rho_{st}(t_0)\hat{N}(\tau)\hat{N}(0)\}, \quad (2)$$

which determine the back-action. We consider lowest non-vanishing order in H_T, i.e. we neglect co-tunneling and higher order effects. Due to the low junction transparency we also neglect logarithmic renormalization effects [22]. In this approximation the rates for the quasiparticle processes involved are given by the usual current-voltage characteristics for tunneling between two superconductors[23], where the effective voltage is given by the energy gain in the process.

Assuming that the SSET is the strongest noise source coupled to the two-level system (TLS) the rate for relaxing/exciting the TLS is given by[6, 24, 25]:

$$\Gamma_{\downarrow/\uparrow} = \alpha \frac{e^2}{\hbar^2} S_V(\pm\Delta E/\hbar) \quad (3)$$

where α is a dimensionless coupling constant and ΔE is the energy splitting between the two states in the TLS. For the SCB $\alpha = \frac{C_c^2}{C_b^2}\frac{E_J^2}{\Delta E^2}$. To describe population inversion we calculate the polarization of the TLS in the steady state, i.e. $\Delta P = P_1^{TLS} - P_0^{TLS} = (\Gamma_\uparrow - \Gamma_\downarrow)/(\Gamma_\uparrow + \Gamma_\downarrow)$. The total back-action induced mixing rate $\Gamma_\uparrow + \Gamma_\downarrow$ is proportional to the symmetrized noise $S_V^{sym}(\Delta E/\hbar) = S_V(\Delta E/\hbar) + S_V(-\Delta E/\hbar)$. If this rate is small, then other noise sources may determine the steady-state of the two-level system. To see where the back-action is strong we also study $S_V^{sym}(\Delta E/\hbar)$.

Results To be specific we now consider a symmetric SSET with the same parameters as in reference [20], i.e. with charging energy equal to half of the superconducting energy gap, $E_C = \Delta_S \approx 25$meV, $C_L = C_R$, and $R_T = R_L \approx 50$kΩ, using the Ambegaokar-Baratoff value for the Josephson energy, $E_J^L = E_J^R \approx 16\mu$eV. We choose a level splitting of the TLS of $\Delta E = 0.1\Delta_S$. To describe the main features in the current through the SSET and the noise in this regime it is enough to take the five lowest-energy charge states into account. Numerically we check that including 7 states does not change the result. In Fig. 2 we show the numerical results for the DC current through the SSET, the symmetrized noise $S_V^{sym}(\Delta E/\hbar)$ and the polarization of the TLS ΔP, as a function of SSET gate voltage and driving bias.

Analytical Results At specific bias points in the sub-gap regime, where specific JQP processes are known to dominate, one may calculate $S_V(\omega)$ analytically. The main approximation is to consider all involved quasiparticle transition rates equal and frequency independent. For frequencies not too high compared to the superconducting energy gap ($\hbar\omega < \Delta_S$) this is a reasonable approximation. Keeping track of the density matrix of up to 5 charge states, and the coherence in two resonant pairs, leads to inverting matrices of dimension up to 9×9. Using Mathematica we obtain analytical results, which are presented below at appropriate places. For the DJQP cycle, in the proper limit, this procedure reproduces the formulas for $S_V(\omega)$ derived by Clerk et al[20].

Simple JQP resonances The Coulomb energy of the charge state with N extra electrons on the island is $E_N = E_C(N - n_x)^2$. Along the lines $eV_{ds} = E_{\pm 2} - E_0 = 4(1 \mp n_x)E_C$ in Fig. 2 the charge states $N = 0$ and $N = \pm 2$ have equal energy, including the work $2(eV_{ds}/2)$ done by the bias voltage across the L/R junction. (Since $C_L = C_R$ the voltage divides equally over the junctions.) Due to the Josephson coupling Cooper pairs tunnel resonantly[26] back and forth across the L/R junction. For $eV_{ds} > 3E_C$ the resonantly tunneled Cooper pair may decay into two quasiparticles tunneling across the opposite (R/L) junction, thus completing the simplest JQP cycle[1], which effectively transports two electrons through the SSET, see Fig. 3a. Approximating the two involved quasiparticle rates with a single frequency-independent rate $\Gamma/\hbar$ and defining the energy gain for the Cooper-pair tunneling as $\delta_\pm = eV_{ds} - 4(1 \mp n_x)E_C$, the population polarization close to a single resonance, but not close to both is

$$\Delta P = \frac{8\delta_\pm \Delta E}{\Gamma^2 + 4(\delta_\pm^2 + \Delta E^2)}, \quad (4)$$

with maximum $\max\{\Delta P^{JQP}\} = 2\Delta E/\sqrt{\Gamma^2 + 4\Delta E^2}$ located at $\delta_\pm = \sqrt{(\Gamma/2)^2 + \Delta E^2}$. We see that for positive $\delta_\pm$ the population is inverted, and that the maximum inversion grows with increased TLS level splitting ΔE.

Qualitative explanation of the PI Without interaction with the TLS the effective Cooper-pair tunneling rate has a maximum at zero energy gain ($\delta_\pm = 0$)[20, 26]. In a JQP cycle which excites the TLS the effective energy gain

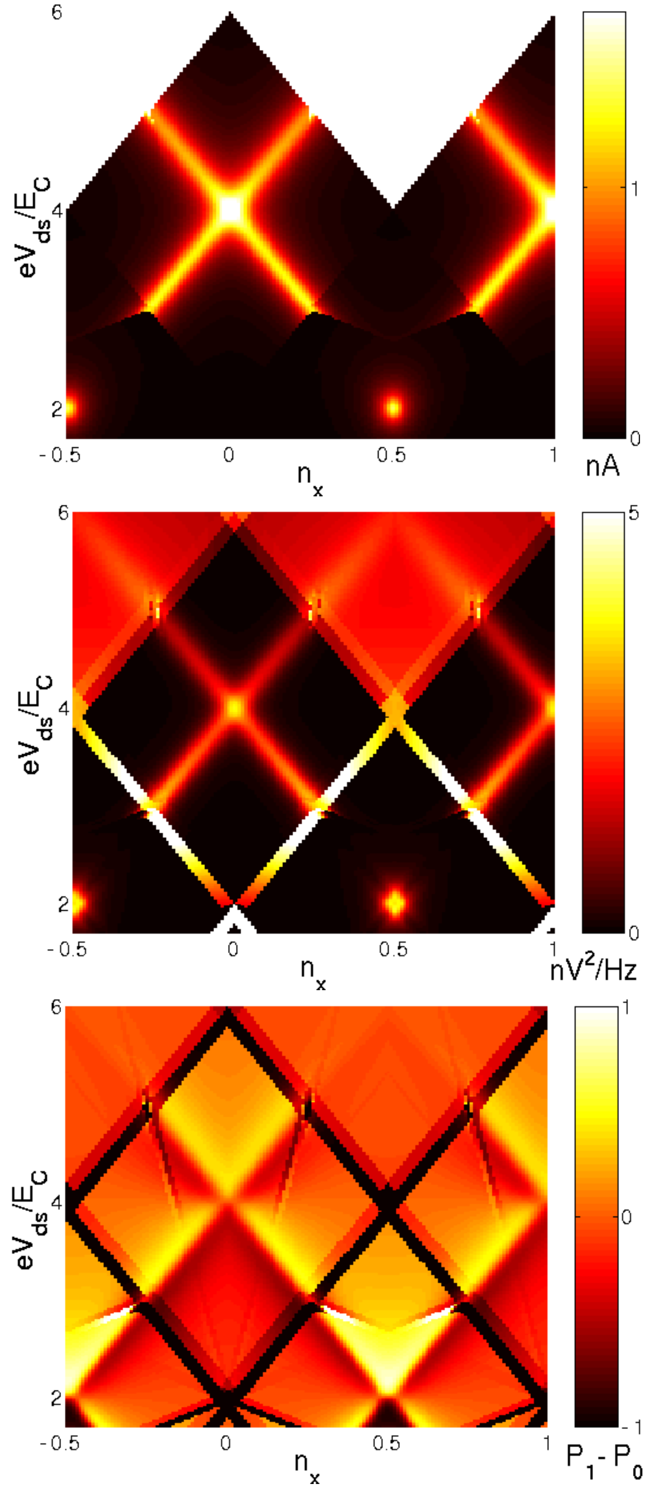

FIG. 2: Numerical results for a symmetric SSET with $E_C = \Delta_S \approx 25$meV, $R_T = R_L \approx 50$kΩ and $E_J^L = E_J^R \approx 16\mu$eV. The level splitting in the two-level system is $\Delta E = 0.1\Delta_S$. Top panel - DC current through the SSET. Middle panel - Total noise $S_V^{sym}(\Delta E/\hbar)$. Lower panel - TLS population polarization $\Delta P = P_1 - P_0$.

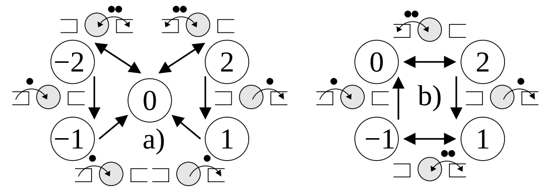

FIG. 3: a) Two simple JQP processes in parallel. The Cooper-pair tunneling in the left/right cycle is resonant along the line $eV_{ds} = 4(1 \pm n_x)E_C$. b) The Double JQP cycle, resonant around $eV_{ds} = 2E_C$ and $n_x = 0.5$.

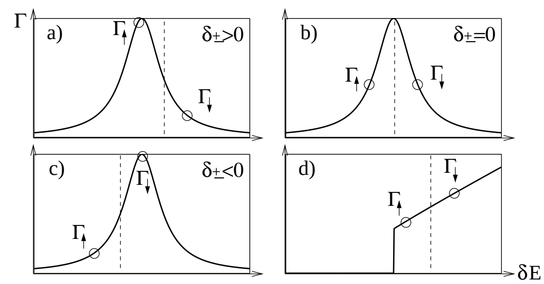

FIG. 4: Schematic figure of effective tunneling rates for Cooper pairs (a-c) and quasiparticles (d) in the SSET, as function of the energy gain δE in the process. The vertical dashed lines denote the energy gain without transitions in the TLS, which in (a-c) is $\delta_\pm$. $\Gamma_{\uparrow/\downarrow}$ indicate tunneling rates with simultaneous excitation/relaxation of the TLS, where the energy gain is shifted by ΔE.

is lowered to $\delta_\pm - \Delta E$, see Fig. 4. For positive $\delta_\pm$ this is closer to resonance (zero) than $\delta_\pm + \Delta E$, which is the gain for the relaxing cycle. Thus the exciting cycle runs faster than the relaxing one, causing a PI of the TLS. As for the quasiparticle tunneling rates they monotonously increase with increasing energy gain (see Fig. 4d). Thus PI will never occur above the threshold voltage, where the current is carried only by quasiparticles. There the relaxing (pure quasiparticle) cycle will always run faster than the exciting one.

Two JQP cycles in parallel Close to $eV_{ds} = 4E_C$ and $n_x = 0$, Cooper-pair tunneling across both the left and right junctions are resonant simultaneously. Here the two JQP cycles run in parallel, see Fig. 3a, sharing the charge state $N = 0$. From the numerical results we deduce that the maximum population inversion is not larger at this crossing compared to the single resonances. Again approximating all involved quasiparticle rates with a single Γ, and now also taking the relevant limit $E_J \ll \Gamma$, we find along the vertical symmetry line $\delta_+ = \delta_- = \delta = eV_{ds} - 4E_C$ and $n_x = 0$, that the population is indeed given by Eq. (4). Along the horizontal line $eV_{ds} = 4E_C$ we have $\delta_+ = -\delta_-$ giving $\Delta P = 0$.

The Double JQP cycle Around $eV_{ds} = 2E_C$ and $n_x = 0.5$, Cooper-pair tunneling between the charge states $N = -1$ and $N = 1$ across the right junction and between $N = 0$ and $N = 2$ across the left junction

are both resonant processes. This is where the Double JQP process[20] occurs, transporting 3e in each cycle, see Fig 3b. The population polarization here is remarkably similar to the one around the simple JQP crossing, except that here it has a real maximum along the symmetry line $n_x = 0.5$. For $E_J \ll \Gamma$ the maximum is again located at $\delta = \sqrt{(\Gamma/2)^2 + \Delta E^2}$, where now $\delta = eV - 2E_C$, and the value is

$$\Delta P_{max}^{DJQP} = \frac{4\Delta E\sqrt{\Gamma^2 + 4\Delta E^2}}{\Gamma^2 + 8\Delta E^2}. \tag{5}$$

Strong Relaxation - No Current For the SSET back-action to be important compared to other noise sources the total noise $S_V^{sym}(\Delta E/\hbar)$ should be large. In the middle panel of Fig. 2 we see that the noise is large where there is substantial current, with one exception: The noise is also large below the lines $eV_{ds}/2 = E_{\pm 1} - E_0 = (1 \pm 2n_x)E_C$, where the width of the region is set by the TLS level splitting ΔE. Here a strong relaxation of the TLS occurs, although there is negligible current. In this bias regime the Cooper-pair tunneling is off resonance, but it still slowly takes the SSET from the lowest energy charge state $N = 0$ to higher energy states. Above these lines the SSET relaxes through quasiparticle tunneling. Thus the SSET mainly occupies its lowest energy charge state $N = 0$. Below these lines this relaxation of the SSET is not energetically allowed, and the SSET ends up with a population of the excited states $N = \pm 1$ of order unity. In the region with strong noise the quasiparticle transition $N = \pm 1$ to $N = 0$ is possible through absorption of the TLS energy, thus causing strong TLS relaxation although there is negligible current flowing through the SSET. In short, the slow off-resonance Cooper-pair tunneling pumps the SSET to an excited state which may only relax through a simultaneous relaxation of the TLS.

Conclusion Using a real-time diagrammatic Keldysh approach we have calculated the back-action noise of a superconducting single-electron transistor, biased in the subgap regime. We find that population inversion of the measured two-level system is possible when the JQP cycle which gives energy to the TLS is closer to resonance than the JQP cycle which takes energy from the TLS. We also find regions where the relaxation of the TLS is strong, although there is negligible current flowing through the SSET, see Fig. 5.

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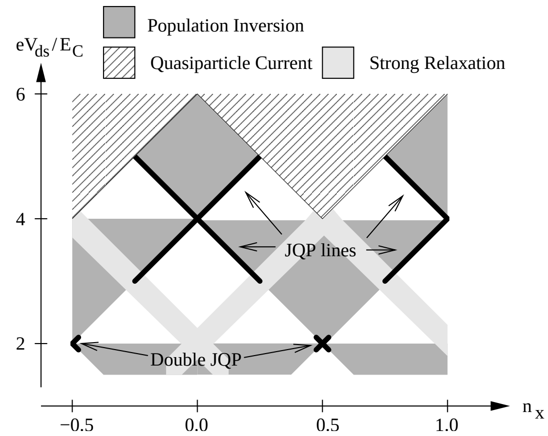

FIG. 5: The dark grey indicates for which SSET bias PI of the TLS occurs. Thick black lines indicate substantial subgap current through the SSET. The light shaded areas indicate where strong TLS relaxation occurs, although there is negligible current running through the SSET.

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